On a Network Creation Game

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Context

The internet has over 12,000 autonomous systems
Everyone picks their own upstream and/or peers
AT&T wants to be close to everyone else on the network, but doesn’t care about the network at large
Question:

What is the “penalty” in terms of poor network structure incurred by having the “users” create the network, without centralized control?
In this talk we...

Introduce a simple model of network creation by self-interested agents
Briefly review game-theoretic concepts
Talk about related work
Show bounds on the “price of anarchy” in the model
Discuss extensions and open problems we believe to be relevant and potentially tractable.
A Simple Model

N agents, each can buy (undirected) links to a set of others \(s_i\)

One agent buys a link, but anyone can use it

Undirected graph \(G\) is built

Cost to agent:

\[
c(i) = \alpha \cdot |s_i| + \sum_j d_G(i, j)
\]

- Pay \(\alpha\) for each link you buy 
  \((\alpha\) may depend on \(n\))
- Pay $1 for every hop to every node
Example

(Convention: arrow from the node buying the link)
It looks like you're designing a network model. Would you like to...

- Obtain an accurate model of how the real Internet behaves.
- Capture all types of phenomena that would exist in such a dynamic system.
- Use a simple model to make a first attempt at understanding a behavior in a broad class of environments.
Definitions

Social cost: \( C(G) = \sum_i c(i) \)

The simplest notion of “global benefit”

Social optimum: combination of strategies that minimizes the social cost

“What a benevolent dictator would do”

Not necessarily palatable to any given agent
Definitions: Nash Equilibria

Nash equilibrium: a situation such that no single player can change what he is doing and benefit

A well-studied notion of “stability” in games, but not uncontroversial

Presumes complete rationality and knowledge on behalf of each agent

Not guaranteed to exist, but they do for our model
Set $\alpha=5$, and consider:
Definitions: Price of Anarchy

Price of Anarchy (Koutsoupias & Papadimitriou, 1999): the ratio between the worst-case social cost of a Nash equilibrium network and the optimum network.

We bound the worst-case price of anarchy to evaluate “the price we pay” for operating without centralized control.
Related Work

Anshelevich, et al. (STOC 2003)

Agents are “global” and pick from a set of links to connect between their own terminals

Results concern the “optimistic price of anarchy” (with best-case Nash equilibria)

A body of similar work on social networks in the econometrics literature (e.g. Bala&Goyal 2000, Dutta&Jackson 2000)
Our Results

Complete characterization of the social optima

Lower and upper bounds on the price of anarchy, constant in $n$, not tight in $\alpha$

A tight upper bound contingent on an experimentally-supported conjecture
Social optima

When $\alpha < 2$, any missing edge can be added at cost $\alpha$ and subtract at least 2 from social cost

When $\alpha \geq 2$, consider a star. Any extra edges are too expensive.
Equilibria: very small $\alpha (<2)$

For $\alpha < 1$, the clique is the only N.E.
For $1 < \alpha < 2$, clique no longer N.E., but the diameter is at most 2; else:

Then, the star is the worst N.E., can be seen to yield P.o.A. of at most $4/3$
General Upper Bound

Assume $\alpha > 2$ (the interesting case)

Lemma: if $G$ is a N.E., $d_G(i, j) < 2\sqrt{\alpha}$

Generalization of the above:

\[-(d-5) \quad -(d-3) \quad -(d-1) = - (d^2)\]
A counting argument then shows that for every edge present in a Nash equilibrium, \((\sqrt{\alpha})\) others are absent.

Then:

\[
C(G) \leq \alpha \cdot O\left(n^2 / \sqrt{\alpha}\right) + n(n - 1) \cdot 2\sqrt{\alpha} \\
= O\left(\sqrt{\alpha} n^2\right)
\]

\(C(\text{star}) = (n^2)\), thus P.o.A. is \(O(\sqrt{\alpha})\).
A Lower Bound

An outward-directed complete k-ary tree of depth $d$, at $\alpha=(d-1)n$:

Infinite penalty for dropping existing links
No new link can bring you more than $(d-1)$ closer to other nodes on average
A Lower Bound

An outward-directed complete k-ary tree of depth d, at $\alpha=(d-1)n$:

Can’t benefit from moving your existing links (the center of each subtree is the optimal site to link to)
A Lower Bound

An outward-directed complete k-ary tree of depth d, at $\alpha = (d-1)n$: 

Benefit from adding several links is convex (net gain $\leq \Sigma$ individual gains), so won’t create several new ones either
A Lower Bound

An outward-directed complete k-ary tree of depth $d$, at $\alpha = (d-1)n$:

For large $d$, $k$, the price of anarchy approaches 3 asymptotically, so $3-\epsilon$ is a lower bound for any $\epsilon > 0$.
So what sorts of equilibria do exist?
Experimental Approach 1

“Simulation”:
Take a random \((G_{n,p})\) graph, iteratively have each agent re-optimize strategy until stable
But\(_1\): no guarantee of convergence
(although converges in practice)
But\(_2\): each iteration is coNP-hard (simple reduction from Dominating Set)

For \(\alpha > 2\), only trees observed, most often stars
Experimental Approach 2

Application of the Feynman Problem-Solving Algorithm:

- Write down $n$
- Think really hard
- Write down a non-tree Nash equilibrium

Third step consistently fails

Sole exception: the Petersen graph for $\alpha < 4$, but still transient
Trees

Conjecture: for $\alpha > \alpha_0$, some constant, all Nash equilibria are trees

Benefit: a tree has a center (a node that, when removed, yields no components with more than $n/2$ nodes)

Given a tree N.E., can use the fact that no additional nodes want to link to center to bound the depth and show that the price of anarchy is at most 5
Discussion

The price tag of decentralization in network design appears modest
not directly dependent on the size of the network being built

The Internet is not strictly a clique, or a star, or a tree, but often resembles one of these at any given scale

Many possible extensions remain to be explored
Some Possible Extensions

What if agents collaborate to create a link?

- Each node can pay for a fraction of a link;
- Link built only if total “investment” is $\geq 1$

May yield a wider variety of equilibria

Stars are efficient for hop distances, but problematic for congestion

What happens when agent costs are penalized for easily-congestible networks?
Some More Possible Extensions

Most agents don’t care to connect closely to everyone else

What if we know the amount of traffic between each pair of nodes and weight the distance terms accordingly?

If \( n^2 \) parameters is too much, what about restricted traffic matrices?

Prevent “perfect blackmail” by making the penalty for complete disconnection large but finite?
Even More Possible Extensions

Charge nodes for Vickrey payments? (from FPSS 2002)

Introduce time?

Network develops in stages as new nodes arrive

Assume equilibrium state is reached at every stage
Directions for Future Work

Proof of tree conjecture
Price of anarchy in the above models
Other points on the spectrum between dictatorship and anarchy?
Measurements to assess applicability to existing real systems
Questions?