CS 70 SPRING 2007 — DISCUSSION #9

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1. Administrivia

(1) Course Information
- Homework 7 is posted on the website, and is due on Tuesday, March 20, at 2:30pm
- There will be a problem set assigned next Tuesday and due the Tuesday after spring break. It will not be any longer than usual; get started early if you don't want it to get in the way of your spring break.
- Mark your calendars: Midterm 2 will be on Tue 04/10

2. Airline combinatorics

Exercise 1. (*Warmup*)

(1) Given n cities, how many ways are there to set up m (one-way) airline routes between them, assuming no two routes connect the same pair of cities? (Without distinguishing routes)
(2) What if we now assign m pilots, one to each route, how many total arrangements of routes and pilots are there?
(3) What if we allow multiple routes from the same origin to the same destination?
(4) What if there are $k < m$ undercover air marshals that can be assigned to accompany pilots, at most one per pilot? How many total arrangements of routes, pilots, and air marshals are there?
(5) What if there may be more than 1 marshal per pilot? (Yes, this allows for absurd possibilities.)

3. Events, sets, and logic

You may've noticed that there's some similarity between set notation and logic notation: $A \cup B$, the union of sets, looks somewhat like $P \vee Q$, the disjunction of propositions (a.k.a. "or"). $A \cap B$, the intersection of sets, looks somewhat like $P \wedge Q$, the conjunction of propositions (a.k.a. "and"). When you think about events as subsets of your probability space, the connection should be clear.

Example 2. You toss a coin and roll a die. Let Ω be the space of possible outcomes ($\Omega = \{T, H\} \times \{1, 2, 3, 4, 5, 6\}$). Let A be the event that coin comes up tails, and $P_A(\omega)$ be the corresponding logical proposition, set to true iff the coin comes up tails in outcome ω. Let B be the event that the die rolls a number higher than 2, and $P_B(\omega)$ be the corresponding logical proposition. The event $A \cup B$ is a set of outcomes ω for which either $P_A(\omega)$ or $P_B(\omega)$ holds. The event $A \cap B$ is a set of outcomes ω for which both $P_A(\omega)$ and $P_B(\omega)$ holds. That is:

$$A \cup B = \{\omega | P_A(\omega) \vee P_B(\omega)\}$$
$$A \cap B = \{\omega | P_A(\omega) \wedge P_B(\omega)\}$$

4. Basic properties of probabilities

Exercise 3. (**Negation property**) Prove: $\Pr[\overline{E}] = 1 - \Pr[E]$

Exercise 4. (**Addition property**) Prove: $\Pr[E \cup F] = \Pr[E] + \Pr[F] - \Pr[E \cap F]$

Definition 5. Conditional probability is defined by $\Pr[E|F] = \frac{\Pr[E \cap F]}{\Pr[F]}$.

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Exercise 6. Notice that the above implies that $\Pr[E \cap F] = \Pr[E|F] \times \Pr[F]$. So, let's resolve the lingering question from lecture — when can you multiply the probabilities of events to get the probability that both happen? That is, when is $\Pr[E \cap F] = \Pr[E] \times \Pr[F]$?

Exercise 7. (**Bayes' Rule**) Prove: $\Pr[E|F] = \frac{\Pr[F|E]\Pr[E]}{\Pr[F]}$.

5. SET!

For the rest of the section, we'll go back to talking about the game SET! from last week's section. If you need a refresher on the rules, see the last page of the handout.

Exercise 8. Suppose you draw 3 cards at random from the deck, without replacement. What is the probability that they make up a SET?

Exercise 9. What is the probability that a card drawn at random from the deck is green, or has diamonds, or is non-shaded?

Exercise 10. You draw two cards from the SET deck and notice they're both squiggles, one red, and one green. Given that information, what's the probability that the next card you draw will make a SET with the first two?

6. Challenge problems

Exercise 11. There are plenty more where these came from, just ask.

(1) What is the probability that the first 12 cards you lay down do not contain a SET?
(2) Can we apply that result to some 12 cards that are on the table mid-way through a game of SET!?
(3) (Relatively easy) Beat Alex at a game of SET after class.

7. Appendix: Rules of SET!™

SET! is played with a special deck of cards that look like those in Figures 1–4.

Each card has 4 properties: shape, shading, number, and color (not shown here). Each property can have one of 3 values:

- shape can be oval, diamond, or squiggle
- shading can be empty, filled, or shaded
- number can be 1, 2, or 3
- color can be green, red, or purple

Each possible card occurs in the deck exactly once, so there're a total of $3 \times 3 \times 3 \times 3 = 81$ cards.

The objective of the game is to find "SETs", defined as groups of 3 cards, such that each property "works"; to "work", a property must be all the same or all different. Figures 1 and 2 show examples of SETs. Figures 3 and 4 show examples of non-sets.

The game is played by laying out 12 cards on a table, and every player looking for SETs. Anyone who finds a SET says, "SET!", picks up those 3 cards, and the dealer adds 3 more cards.

When everyone agrees that there's no SET on the table, the dealer adds 3 more cards. The total number of cards on the table goes down to 12 at the first opportunity (i.e. a dealer never adds cards when there're 12 cards on the table already, unless everyone agrees there's no SET).

Theorem 12. *Given two cards, show that there exists exactly one card that can be added to them to make a set.*

7.1. Mini-SET!.

A reduced version of the game, which we played last week, involves "removing" one of the properties (color) – that is, splitting the deck into 3 27-card decks — a red deck, a green deck, and a purple deck, and playing with just one of the decks. In that case, instead of putting down 12 cards, the dealer puts down 9.

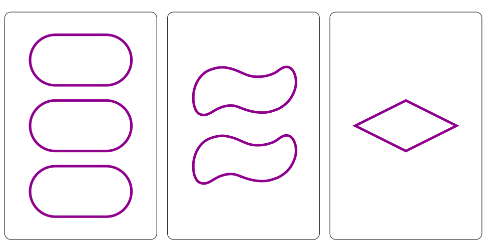

Figure 1. A SET with same color, same shading, different shape, different number

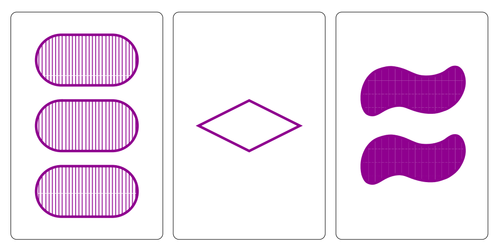

Figure 2. A SET with same color, different shading, different shape, different number

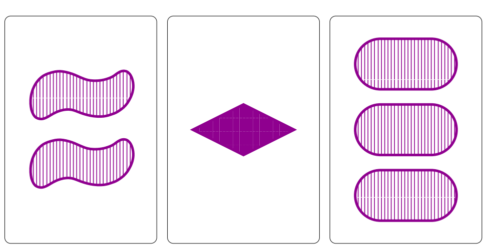

Figure 3. Not a SET — shading doesn't work

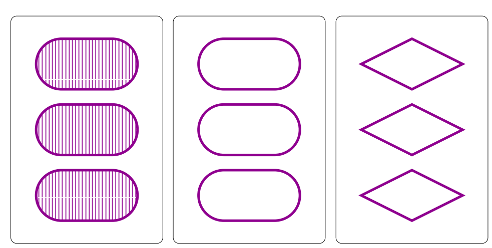

Figure 4. Not a SET — shading and shape don't work

(*For lack of a color copier, assume all of the above cards are* **purple**. *Thus, color always works*)