CS 70 SPRING 2007 — DISCUSSION #8

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1. Administrivia

(1) Course Information
- Homework 6 is posted on the website, and is due, this time only, on *Wednesday*, March 14, at 2:30pm
- MT1 stats: mean 43.0/54, stdev 7.3, median 45 (*good job!*)
- Mark your calendars: Midterm 2 will be on Tue 04/10

2. Hamiltonian paths in $K_{m,n}$

As you may've seen from the midterm review problems, a "complete bipartite graph", denoted $K_{m,n}$, is a graph with $m + n$ nodes that has m "male" nodes and n "female" nodes, and an edge for every possible heterosexual couple (that is, every node on the male "side" is connected to every node on the female "side"). Figure 1 shows $K_{2,3}$.

Exercise 1. Does $K_{2,3}$ have a Hamiltonian path? How many different Hamiltonian paths does it have?

Exercise 2. In general, how many Hamiltonian paths does $K_{m,n}$ have? For simplicity, assume $m \geq n$.

3. Hypercube combinatorics

3.1. **Hamming weights.** The *Hamming weight*[1] of a bitstring s, denoted $|s|_1$, is the number of 1's the bit string contains; so $|01101|_1 = 3$ and $|0000|_1 = |000000|_1 = 0$; this concept comes up all over the place in CS (including CS70 HW6). As with many concepts about hypercubes, we can visualize it particularly well for the 3-dimensional cube: orient the cube as shown in Figure 2, and nodes with the same Hamming weight will lie in the same "horizontal plane". It's often useful to draw the graph structure of higher-dimensional hypercubes using the same approach; Figure 3 shows the 4-dimensional hypercube laid out the same way.

Exercise 3. How many n bit strings are there of Hamming weight w?

Date: March 10, 2007.

The author gratefully acknowledges the TA's of CS70 Past for the use of their previous notes: Amir Kamil, Chris Crutchfield, David Garmire, Lorenzo Orecchia, and Ben Rubinstein. Their notes form the basis for this handout.

[1] No strings were "hammed" in the making of this term. It's named after its inventor, Richard Hamming.

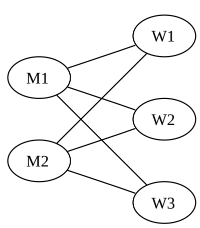

FIGURE 1. $K_{2,3}$

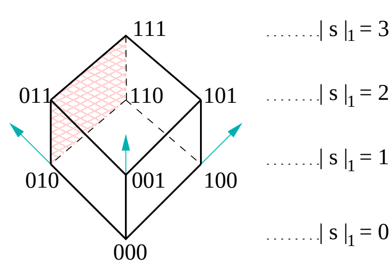

FIGURE 2. 3-cube with Hamming weights

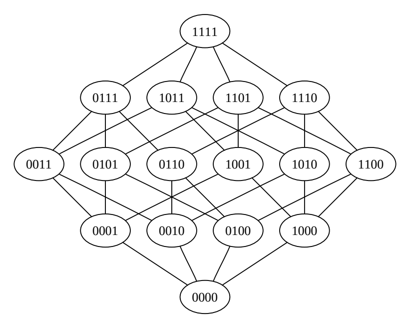

FIGURE 3. 4-D hypercube graph

3.2. **The many faces of hypercubes.** A 3-dimensional cube has 8 vertices, 12 edges, and 6 faces. Each of these objects can be described with a simple bitstring-based definition. For instance, the edge shown as a vertical dashed line in Figure 2 "contains" all nodes matching the pattern "11$*$" (i.e., $x = 1$, $y = 1$, and z is unspecified). The shaded face is represented by the pattern "$*1*$". In general, if k dimensions aren't specified (in a 3-cube, $k = 1$ for an edge, $k = 2$ for a face, $k = 0$ for a vertex), while the other $n - k$ dimensions are fixed, the resulting object is an k-dimensional "subhypercube".

For arbitrary n, these subhypercubes are called k-faces. So, in a 3-cube, a "face" as we know it is a "2-face" and an edge is a "1-face" (technically, a vertex is a "0-face", though this is rarely used). A 4-cube would have 1-faces (edges), 2-faces, and 3-faces.

Exercise 4. How many edges (1-faces) does a 4-dimensional hypercube have? How about 2-faces? 3-faces?

Exercise 5. How many k-faces does an n-dimensional hypercube have?

4. SET!

For the rest of the section, we'll focus on a game called "SET!". It's played with a special deck of cards that look like those in Figures 4–7.

Each card has 4 properties: shape, shading, number, and color (not shown here). Each property can have one of 3 values:

- shape can be oval, diamond, or squiggle
- shading can be empty, filled, or shaded
- number can be 1, 2, or 3
- color can be green, red, or purple

Each possible card occurs in the deck exactly once.

Exercise 6. How many cards are there in the deck?

The objective of the game is to find "SETs", defined as groups of 3 cards, such that each property "works"; to "work", a property must be all the same or all different. Figures 4 and 5 show examples of SETs. Figures 6 and 7 show examples of non-sets.

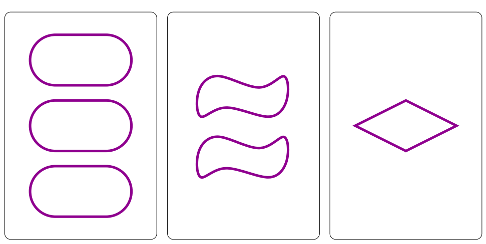

FIGURE 4. A SET with same color, same shading, different shape, different number

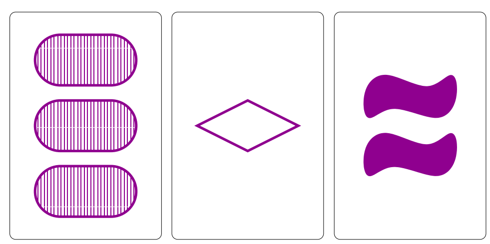

FIGURE 5. A SET with same color, different shading, different shape, different number

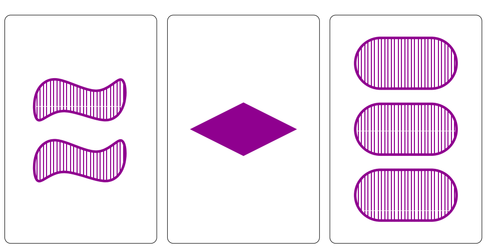

FIGURE 6. Not a SET — shading doesn't work

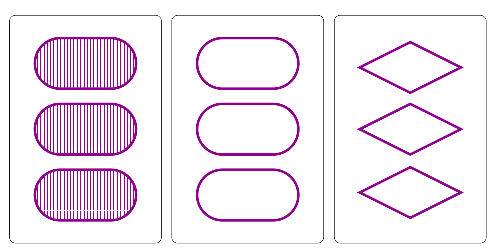

FIGURE 7. Not a SET — shading and shape don't work

Exercise 7. Given two cards, show that there exists exactly one card that can be added to them to make a set.

Exercise 8. Use the exercise 7 to count the number of possible different sets.

Exercise 9. How many different 3x3 "magic squares of SET cards" are there? A "magic square" means that any vertical line, horizontal line, or diagonal, constitutes a set.

5. Challenge problems

Exercise 10. There are plenty more where these came from, just ask.

(1) (Somewhat hard) Find 20 cards such that there are no sets among them.
(2) (Much harder) Prove that any 21 cards from the deck must contain a set.
(3) (Relatively easy) Beat Alex at a game of SET after class.