CS 70 SPRING 2007 — DISCUSSION #7

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1. Administrivia

(1) Course Information
- **Today (Friday, Mar 2nd), 5pm-7pm**, there'll be a midterm review session in 306 Soda. We'll go over any midterm practice problems, and whatever other questions you guys bring in.
- **Midterm 1 will be on Tuesday, March 6th, in class**. You'll be allowed **one single-sided** $8.5'' \times 11''$ **page of notes**.
- Next homework will be released on Tuesday 3/06, and will be due a week after that.
- HW4 stats: mean 13.4/22, stdev 8.1, median 16.5

2. Eulerian Graphs

Theorem 1. *An undirected (respectively directed) graph has an Eulerian tour iff every vertex has even degree (respectively iff every vertex has equal in- and out-degree).*

Exercise 2. Suppose an undirected graph G has an Eulerian tour. How can you "convert" G into a directed graph by assigning a direction to each edge in such a way that the resulting directed graph also has an Eulerian tour?

Exercise 3. What is an easy procedure of adding edges to use to make an (undirected) complete binary tree to have an Eulerian Tour?

3. Induction on graphs

Exercise 4. Let's look at undirected graphs with an even number $2k$ of vertices that are "triangle-free". A graph (V, E) is said to be triangle-free if there are no 3 nodes $\{A, B, C\} \in V$ such that every pari of them is connected by an edge ($\{\{A, B\}, \{B, C\}, \{A, C\}\} \in E$).

(1) Use induction to prove that a triangle-free graph with $2k$ vertices and no triangles has at most k^2 edges. *Hint*: The inductive step needs to be a bit unusual — for an arbitrary $2(k+1)$-node triangle-free graph, *first* observe that if it has no edges whatsoever, we're done; otherwise take an arbitrary edge $\{A, B\}$ and use the inductive assumption on the graph defined by removing A and B.
(2) Give an example where this upper bound is achieved.

4. Hypercubes

Theorem 5. *A graph is called a "1-expander" graph if in order to separate n vertices from the rest of the graph you must cut n edges (for $n < \frac{1}{2}|V|$).*

Theorem 6. *In H_n (the n-hypercube) to isolate any set S of vertices $|S| \leq 2^{n-1}$ you need to cut $|E_S| \geq |S|$ edges.*

Exercise 7. If you have two hypercubes of the same size, how many edges do you need to add between them in order to make the graph a 1-expander?

Exercise 8. When can a hypercube also have an Eulerian Tour?

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5. de Bruijn sequences

Exercise 9. Occasionally, it's useful to have logic devices capable of handling "trits" (the ternary, aka base-3, equivalent of the binary bits). Are de Bruijn sequences guaranteed to exist in base-3? In base-n?

6. Challenge problem

Exercise 10. Prove the *Bondy-Chvátal Theorem*: Given an undirected graph $G = (V, E)$ which has two vertices, u, and v, whose degrees add up to at least the number of vertices ($\deg u + \deg v \geq |V|$), and which are non-adjacent, then G has a Hamiltonian cycle if and only if the graph G' obtained from G by adding the edge $\{u, v\}$ has a Hamiltonian cycle. (Which direction of the "if and only if" is easy?)