CS 70 SPRING 2007 — DISCUSSION #2

ALEX FABRIKANT

1. Administrivia

(1) Course Information
  - The first homework is **due January 30th** at 3pm in 283 Soda Hall, and is posted on the course website
  - You are encouraged to work on the homework in groups of 3-4, but write up your submission *on your own*. Cite any external sources you use.

2. Biconditional Proofs

Last week's lecture introduced a number of types of proofs, including direct proofs and proof by contraposition which both aim to prove a statement of the form $P \to Q$. Often our goal will *additionally* be to prove the converse $Q \to P$. The proposition that the first *and* the second holds, $(p \to q) \wedge (q \to p)$, is written as $p \leftrightarrow q$. This operator is called the biconditional (or bidirectional implication). While $p \to q$ can be thought of as "If p, then q", $p \leftrightarrow q$ can be thought of "p if and only if q."

Theorem 1. *n is odd iff n^2 is odd, for each $n \in \mathbb{N}$.*

Exercise 2. Consider Theorem 1.

(i) Begin by proving the forward direction (n odd implies n^2 odd).
(ii) Carefully complete the proof of the theorem with a simple modification to part (i).
(iii) Appeal to the equivalence of an implication and its contrapositive to prove the corollary[1] that "n is even iff n^2 is even, for each $n \in \mathbb{N}$.

3. How not to write inductive proofs

Exercise 3. What's wrong with the following write-up of the inductive proof that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$, from class?

Base case: $1 = 1 \cdot (1+1)/2$.

Inductive step:

$$\begin{aligned}
\sum_{i=1}^{n+1} i &= \frac{(n+1)((n+1)+1)}{2} \\
n+1+\sum_{i=1}^{n} i &= \frac{(n+1)n}{2} + \frac{2(n+1)}{2} \\
n+1+\sum_{i=1}^{n} i &= \frac{(n+1)n}{2} + n + 1 \\
\sum_{i=1}^{n} i &= \frac{(n+1)n}{2}
\end{aligned}$$

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[1] A 'corollary' is a result that immediately follows from a proven result.

4. Algebraic Inductions

Let's try some practice induction problems that look like those covered in lecture this week.

Exercise 4. Prove that $1^2 + 2^2 + 3^2 + \ldots + n^2$ equals $\frac{1}{6}\left(2n^3 + 3n^2 + n\right)$.

Exercise 5. A geometric series is an infinite sum of the form $1 + x + x^2 + x^3 + x^4 + \ldots$ for some real x. Prove that the series' partial sum $1 + x + x^2 + \ldots + x^n$ equals $\frac{x^{n+1}-1}{x-1}$.

5. Non-Algebraic Inductions

Exercise 6. Prove the *transitivity* property of bidirectional implication: given n propositions $A_1, A_2, \ldots, A_n$ for any integer $n \geq 3$, show that $A_1 \leftrightarrow A_2 \leftrightarrow A_3 \leftrightarrow \ldots \leftrightarrow A_n$ entails $A_1 \leftrightarrow A_n$.

Induction is also a powerful proof technique in many geometric problems.

Exercise 7. An *arrangement* of lines in the plane $\mathbb{R}^2$ is a set of lines satisfying the property that no point in the plane $x \in \mathbb{R}^2$ is the intersection of 3 or more lines

(i) Draw some example sets of lines that are/aren't legal arrangements.

An arrangement divides the plane up into *cells* or polyhedral regions that have segments of the arrangement's lines as borders. Two cells are called *neighbors* if they share a border segment (shared border points don't count). A 2-coloring of the cells of an arrangement is an assignment of 'black' or 'white' to each cell such that no neighboring cells share the same color.

(ii) Use induction to prove that an arrangement's cells can always be 2-colored.

□

6. Strong Induction: Sums of Fibonacci & Prime Numbers

Many of you may have heard of the Fibonacci sequence. We define $F_1 = 1, F_2 = 1$, and then define the rest of the sequence recursively: for $k \geq 3$, $F_k = F_{k-1} + F_{k-2}$. So the sequence ends up looking like:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$$

While not all positive integers are Fibonacci (e.g. 4), surprisingly we can express any positive integer as the sum of distinct terms in the Fibonacci sequence.

Theorem 8. *Every positive integer n can be expressed as the sum of distinct terms in the Fibonacci sequence.*

Exercise 9. Informally, why does the basic, "weak", form of induction get stuck if you try to apply it to this problem?

Proof. Let $P(n)$ be the statement that n can be expressed as the sum of distinct terms in the Fibonacci sequence. We begin with the base case $n = 1$. Since 1 is a term in the Fibonacci sequence (namely F_1), then $P(1)$ is true.

Now we proceed to the inductive step. We wish to show that $P(1) \wedge P(2) \wedge \cdots \wedge P(n) \implies P(n+1)$. So assume that $P(1), P(2), \ldots, P(n)$ hold. Now we consider $n + 1$. There are two cases:

(1) $n + 1$ is itself a Fibonacci number.
(2) $n + 1$ is not a Fibonacci number.

If the former holds, then we're done. If the latter holds, then there must exist some positive integer k such that

$$F_k < n + 1 < F_{k+1}.$$

Since $F_k < n + 1$, we may decompose $n + 1$ into $F_k + (n + 1 - F_k)$. But by definition, $(n + 1 - F_k) < n + 1$ so by the inductive hypothesis we know that $P(n + 1 - F_k)$ is true, hence it may be expressed as such:

$$n + 1 - F_k = F_{i_1} + F_{i_2} + \cdots + F_{i_m}$$

where the subscripts are distinct. Moreover, since $n + 1 - F_k < F_k$ (since $n + 1 < F_{k+1}$ implies $n + 1 - F_k < F_{k-1} < F_k$) it is not possible that any of the F_{i_j} could be equal to F_k. Therefore we have

$$n + 1 = F_k + F_{i_1} + F_{i_2} + \cdots + F_{i_m}$$

and $P(n + 1)$ holds. Thus by strong induction, $P(n)$ holds for all $n \geq 1$. □

Exercise 10. Chocolate bars are often rectangular, consisting of $a \times b$ small squares. You can break them along any horizontal or vertical line separating the small squares. You're only allowed to break any one contiguous piece of chocolate at a time.

(1) Consider a chocolate bar with $n \times 1$ small squares. How many breaks does it take to break it down into n small squares?
(2) Now, suppose you have a chocolate bar with n small squares in total, not necessarily all in one row. Perhaps surprisingly, the same formula applies! Use strong induction to prove this.

7. Challenge problem

Exercise 11. Let $a_1, \ldots, a_n$ be positive real numbers. Use (strong) induction to prove that their arithmetic mean (average) is always greater or equal to their geometric mean, $\left(\prod_{i=1}^{n} a_i\right)^{1/n}$.