CS 70 SPRING 2007 — SOLUTION TO EX. 10 FROM SECTION 2

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Exercise 10. Chocolate bars are often rectangular, consisting of $a \times b$ small squares. You can break them along any horizontal or vertical line separating the small squares. You're only allowed to break any one contiguous piece of chocolate at a time.

(1) Consider a chocolate bar with $n \times 1$ small squares. How many breaks does it take to break it down into n small squares?

Solution: Take a look at Figure 1. Any time you break the 6×1 bar, you separate it along one of the dotted red lines. No matter what you do, you'll have to break it along each of those lines separately, thus requiring 5 separate breaks. The same holds for any $n \times 1$ bar — since only one of the "boundaries" between the squares is broken each time, you'll need $n - 1$ breaks to separate it completely.

(2) Now, suppose you have a chocolate bar with n small squares in total, not necessarily all in one row. Perhaps surprisingly, the same formula applies! Use strong induction to prove this.

Solution: Per the problem, we'll use strong induction to prove, for all positive integers n, the statement $P(n)$, "Any chocolate bar consisting of n small squares requires $n - 1$ breaks to break it down all the way into separate small squares."

Base case: if $n = 1$, the chocolate "bar" is just a single small square, so zero breaks are needed.

Inductive step: Suppose, for some specific positive integer n, we know that $P(k)$ holds for all integers k between 1 and n. Consider a bar consisting of $n+1$ squares. $n+1 \geq 2$, so there're multiple small squares in the bar that can still be broken apart in some way. Break the $n+1$-square bar into two pieces in some way (see, e.g. Figure 2). This will form pieces of size a and $b = n + 1 - a$, with both sides having at least one square. Thus, $1 \leq a \leq n$, and $1 \leq b \leq n$. By our inductive hypothesis, we know that $P(a)$ and $P(b)$ holds — that is, $a-1$ breaks will be required to finish breaking the first piece into small squares, and $b-1$ breaks will be required for the second piece. Thus, the total number of breaks needed to break apart the $n+1$-square bar (including the first break that splits it into an a-square bar and a b-square bar) will be $1+(a-1)+(b-1) = 1+a-1+n+1-a-1 = n = (n+1)-1$, so $P(n+1)$ holds as well.

By strong induction, $P(n)$ holds for all positive integers n. □

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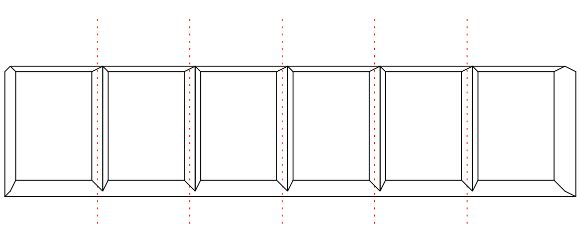

FIGURE 1. The 5 breaks needed to break a 6×1 bar.

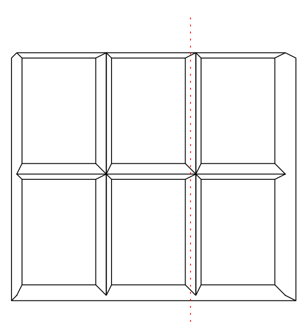

FIGURE 2. The first break for a 3×2 bar, creating pieces of sizes $a = 4$ and $b = 2$.