1. Administrivia

(1) The last optional “supplementary” section on Wednesdays, 4:30-7:00pm (and possibly longer than that), in 320 Soda. Bring any questions whatsoever.
(2) The last homework (homework 12) has been posted, and will be due on Tuesday, 05/08.
(3) We’ll post practice problems for the final on Tuesday; watch the webpage for details about the final review session
(4) The final is on Friday, 05/18 (2 weeks from today), 12:30-3:30pm, at 105 North Gate (school of journalism, immediately southeast of the Hearst & Euclid intersection. You may bring $3 \times 8.5 \times 11$ pages of handwritten notes — and no magnifying glass.

2. Warmup

Exercise 1. Fill in the blanks in Tables 1 and 2 (next page).

3. Chernoff bounds

**Theorem 2** (Chernoff bound). If $X$ is a sum of independent 0/1 (indicator) r.v.’s, and $\mathbb{E}[X] = \mu$, then $\Pr[X > \alpha \mu] \leq e^{\alpha \mu - \mu - \alpha \mu \ln \alpha}$.

**Exercise 3.** A ping packet sent from my currently flaky DSL line to inst.eecs this morning arrives on average in 88 milliseconds, with a standard deviation of 83 milliseconds. Consider the following approaches to writing a script to automatically email my ISP and complain, and decide which bound would work best for each probability:

(1) Email whenever it sees a single ping with delay over 500 milliseconds. Bound the probability that a particular packet will cause a complaint.
(2) Email whenever it sees 90% of a sample of 100 packets take longer than 120 milliseconds. Bound the probability that a particular sequence of 100 packets would trigger this. What would you need to assume?
(3) Email whenever it sees 2 out of 3 consecutive packets take longer than 200 milliseconds. Why does this not get nicely bounded by the same technique as above?

In lecture, we derived Chernoff bounds by applying a Markov bound to $\Pr[\alpha X > \alpha \mu]$, and Luca mentioned that any monotonic, non-negative function would work as well.

**Exercise 4.** What happens if we decide to get an even stronger bound by applying Markov to $\Pr[2^{\alpha X} > 2^{\alpha \mu}]$?

4. Infinites

**Exercise 5.** Decide whether each of these is finite, countably infinite, or uncountable:

(1) The set of all bit strings ($\{\text{emptystring}, 0, 1, 00, 01, \ldots\}$).
(2) The set of all possible C programs.
(3) The set of all possible JPEG images.
(4) The set of all possible “general” (pixel) images, as an assignment of one of 16 million colors to each pair of integer coordinates.

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*Date: May 4, 2007.*

The author gratefully acknowledges the TA’s of CS70 Past for the use of their previous notes: Amir Kamil, Chris Crutchfield, David Garmire, Lorenzo Orecchia, and Ben Rubinstein. Their notes form the basis for this handout.
(5) The set of all possible GPAs (assuming they're not rounded).
(6) The set of all possible Berkeley transcripts.
(7) The set of all possible ways to enumerate the sand grains at Stinson beach (“this one is grain #1, this one is grain #2, etc.”).
(8) The set of all possible ways to assign an integer to all sand grains at Stinson beach.
(9) The set of all possible ways to assign “good” and “bad” to each possible bit string.

5. CHALLENGE EXERCISES

**Exercise 6.** Compare the cardinalities of the following to the cardinalities of $\mathbb{Z}$ and $\mathbb{R}$:

1. $\mathbb{R}^2$ (points in the Euclidean plane)
2. $\mathbb{R}^\mathbb{Z}$ (the number of functions from the integers to the reals; why does this notation make sense as an extension of the $\mathbb{R}^2$ notation?)
3. $\mathbb{R}^\mathbb{R}$ (the number of functions from $\mathbb{R}$ to $\mathbb{R}$)
4. $\mathbb{Z}^\mathbb{R}$ (the number of functions from the reals to the integers)
5. $2^{\mathbb{R}^2}$ (first, figure out what this notation means)
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<tbody>
<tr>
<td>Markov bound</td>
<td>Yes</td>
<td></td>
<td></td>
<td>Definite linear decrease (of $\Pr[X &gt; k\mu]$, as $k$ grows)</td>
<td>If it’s very probable that it’s much higher than mean, then $\Pr[\text{low}]$ can’t be high enough to average out to the mean.</td>
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<tr>
<td>Chebyshev bound</td>
<td>Yes</td>
<td></td>
<td></td>
<td>Definite quadratic decrease (of $\Pr[</td>
<td>X - \mu</td>
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<tr>
<td>Chernoff bound</td>
<td>Yes</td>
<td></td>
<td></td>
<td>Definite exponential decrease (of $\Pr[X &gt; k\mu]$, as $k$ grows)</td>
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<tr>
<td>Law of Large Numbers</td>
<td>Yes</td>
<td></td>
<td></td>
<td>Asymptotically, as $n$ grows, $X$ gets asymptotically close to the mean</td>
<td>If variance is finite, use Chebyshev; else, complicated</td>
</tr>
<tr>
<td>Central Limit Theorem</td>
<td>Yes</td>
<td></td>
<td></td>
<td>Asymptotically, as $n$ grows, distribution of $X$ looks like a normal of width $\sigma/\sqrt{n}$ around $\mu$</td>
<td>Far beyond CS70 scope</td>
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**Table 1.** Summary of major probability theorems from CS70

<table>
<thead>
<tr>
<th></th>
<th>Pros</th>
<th>Cons</th>
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<tbody>
<tr>
<td>Markov bound</td>
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<td>Chebyshev bound</td>
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<tr>
<td>Central Limit Theorem</td>
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**Table 2.** Comparison of probability theorems